Stability of Schwarzschild for the spherically symmetric Einstein–massless Vlasov system

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GR and Hyperbolic PDE Seminar
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Outline of the talk

1. The main result
2. The linear problem
3. The nonlinear difficulties

General Relativity

General relativity is a geometric theory of *gravitation* whose main object of study are the *Lorentzian manifolds* $(\mathcal{M}^{1+n}, g)$ * for which the *Einstein field equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

are satisfied, where $T_{\mu\nu}$ is the *energy momentum tensor of <u>matter</u>*. *Ex

Naturally, we are interested in the *Einstein vacuum equations* (EVE)

$$R_{\mu\nu} = 0. \tag{2}$$

· Minkowski · Schwarzschild · Kerr

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

The stability problem in GR

The dynamic nature of the EVE become apparent when the system is formulated as a *Cauchy problem*.

Theorem (Choquet-Bruhat)

The Einstein vacuum equations are well-posed in Sobolev regularity.

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Question: Is Minkowski/Schwarzschild/Kerr *stable* as solution of the EVE?

Minkowski C-K, L-R

Schwarzschild K-S, DHRT

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Conjecture: The subextremal family of Kerr black holes is stable as solution of the EVE? * Matter and stat solut.

The massless Vlasov equation

Let $(\mathcal{M}^{1+n}, g)$ be a Lorentzian manifold. We introduce a non-negative *distribution function* $f : \mathcal{P} \to [0, \infty)$ which is defined in the manifold

* of matter
* ptcle dst

$$\mathcal{P} := \Big\{ (x,p) \in T\mathcal{M} : g_x(p,p) = 0, \; p_0 > 0 \text{ for every } x \in \mathcal{M} \Big\}. \quad (3)$$

Note that the distribution function is only* supported on *null vectors*. We call $\mathcal{P}$ the *mass-shell*.

$$T_{(x,p)} TM = \mathcal{V}_{(x,p)} \oplus \mathcal{H}_{(x,p)}$$

$$Z = Z^{\mu} e_{\mu}$$

$$Z^{v} = Z^{\mu} \partial_{p^{\mu}}$$

$$Z^{h} = Z^{\mu} e_{\mu} - Z^{\mu} p^{\nu} \Gamma^{\lambda}_{\mu\nu} \partial_{p^{\lambda}}$$

$$\bar{g}(X^h, Y^h) = g(X, Y)$$

$$g(X^h, Y^v) = 0$$

$$g(X^v, Y^v) = g(X, Y)$$

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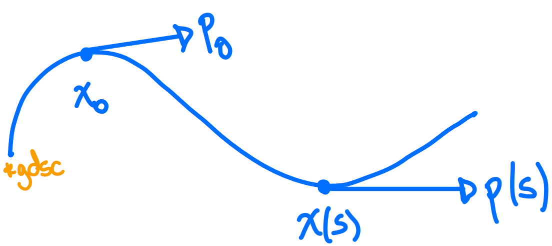
Note that the distribution function is <u>only</u> supported on *null vectors*. We call $\mathcal{P}$ the *mass-shell*. Naturally, we introduce the *massless Vlasov equation* given by

$$p^\alpha \partial_{x^\alpha} f - p^\alpha p^\beta \Gamma^\gamma_{\alpha\beta} \partial_{p^\gamma} f = 0. \tag{4}$$

$$f\big|_\Sigma = f_0$$

Cauchy problem

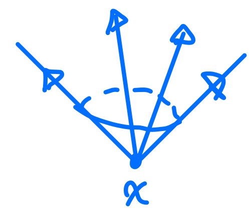


$$f(x(s), p(s)) = f(x_0, p_0)$$

The Einstein–massless Vlasov system

Motivated by the study of *collisionless* many particle systems* in general relativity, we research the Einstein equations coupled to a matter model coming from *kinetic theory*. We define the *energy momentum tensor for massless Vlasov* as

$$T_{\mu\nu}(x) := \int_{\mathcal{P}_x} f p_\mu p_\nu \, \mathrm{dvol}_{\mathcal{P}_x} \,. \qquad (5)$$



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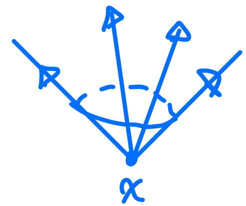


Finally, the *Einstein–massless Vlasov system* (EV) is defined by

* Synge, Jeans

* Collisionless Boltz, Galactic dynam.

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \\ X(f) := p^\alpha \partial_{x^\alpha} f - p^\alpha p^\beta \Gamma^\gamma_{\alpha\beta} \partial_{p^\gamma} f = 0, \end{cases} \tag{6}$$

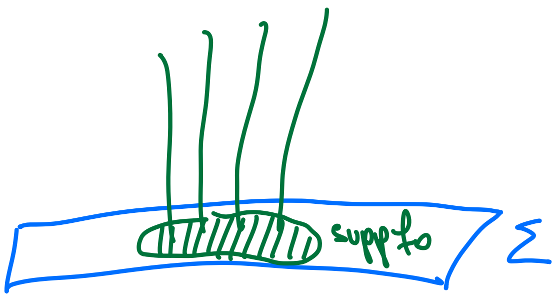
where the initial data is given by $S = (\Sigma, g_0, k_0, f_0)$.

The Einstein–massless Vlasov system II

The Cauchy problem for this matter model defines a mixed hyperbolic -transport type system of nonlinear PDEs.

Theorem (Choquet-Bruhat)

The Einstein–Vlasov system is well-posed in Sobolev regularity.



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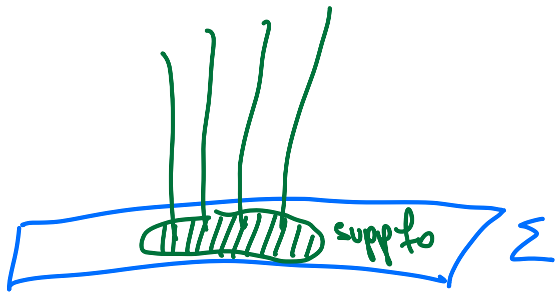

Question: Is Minkowski/Schwarzschild/Kerr stable as solution of the EV system? *EVE?

Literature review

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3. Integrated energy decay for the massless Vlasov equation in slowly rotating Kerr (Andersson-Blue-Joudioux). +no ptw + AB.

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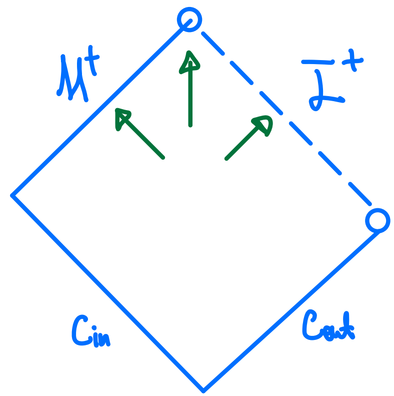
3. Integrated energy decay for the massless Vlasov equation in slowly rotating Kerr (Andersson-Blue-Joudioux).

4. Superpolynomial decay for the massless Vlasov equation in Schwarzschild (Bigorgne). $* r^p$ mth, no ∂_p

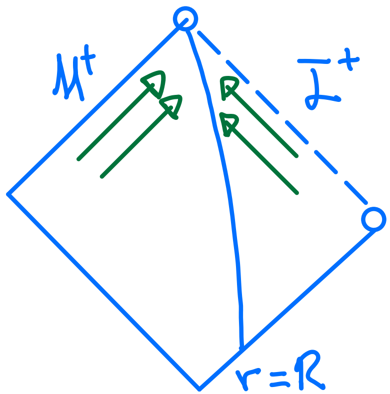
Asymptotic stability of Schwarzschild



Theorem (V.)

The exterior of the Schwarzschild family is asymptotically stable as a solution of the spherically symmetric Einstein–massless Vlasov system. More precisely, for every initial data sufficiently close to Schwarzschild, the resulting solution asymptotes exponentially to another member of the Schwarzschild family.

The main result: linear version



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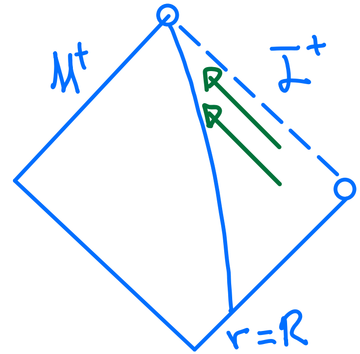


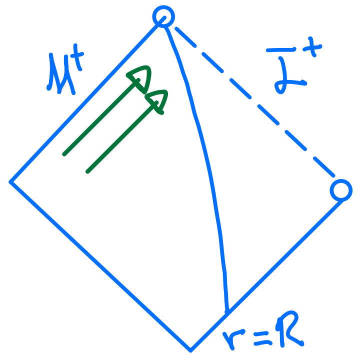
Theorem (Decay of the stress energy momentum tensor)

Let f_0 be a compactly supported initial data for the massless Vlasov equation in Schwarzschild. There exists a positive constant $R > 2M$ such that the solution f of the massless Vlasov equation in Schwarzschild satisfies

$$T_{vv} \leq \frac{C_1}{r^6 \exp(C_2 u)}, \quad T_{uv} \leq \frac{C_1}{r^4 \exp(C_2 u)}, \quad T_{uu} \leq \frac{C_1}{r^2 \exp(C_2 u)}, \tag{7}$$

for all $(u, v) \in \{r \geq R\}$, where C_1 and C_2 are two positive constants depending on f_0, M and R. *∂T dcy, weights *Rlvt cmpts

The main result: linear version



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$$T_{vv} \leq \frac{C_1}{\exp(C_2 v)}, \quad T_{uv} \leq \frac{C_1(1-\frac{2M}{r})}{\exp(C_2 v)}, \quad T_{uu} \leq \frac{C_1(1-\frac{2M}{r})^2}{\exp(C_2 v)}, \tag{8}$$

for all $(u, v) \in \{r \leq R\}$, where C_1 and C_2 are two positive constants depending on f_0, M and R.

The Einstein equations under spherical symmetry

Let $(\mathcal{M}^{3+1}, g)$ be a spherically symmetric spacetime in *double null coordinates* given by

$$g = -\frac{\Omega^2}{2}(du \otimes dv + dv \otimes du) + r^2(u,v) d\gamma_{\mathbb{S}^2}, \tag{9}$$

where Ω and r are two positive functions and γ is the standard metric of $\mathbb{S}^2$ in polar coordinates.

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where Ω and r are two positive functions and γ is the standard metric of $\mathbb{S}^2$ in polar coordinates. We introduce the *spherically symmetric Einstein–massless Vlasov system* by

$$\begin{cases} \partial_u \partial_v r & = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + 4\pi r T_{uv}, \\ \partial_u \partial_v \log \Omega & = \frac{\Omega^2}{4r^2} + \frac{\partial_u r \partial_v r}{r^2} - 4\pi T_{uv} - \pi \Omega^2 g^{AB} T_{AB}, \\ \partial_u(\Omega^{-2}\partial_u r) & = -4\pi r T_{uu} \Omega^{-2}, \\ \partial_v(\Omega^{-2}\partial_v r) & = -4\pi r T_{vv} \Omega^{-2}, \end{cases} \tag{10}$$

where T_{uu}, T_{uv} and T_{vv} are components of the energy momentum tensor.

The Hawking mass

We introduce a key pointwise quantity for the spherically symmetric Einstein equations: *the Hawking mass*. We define the Hawking mass as the real-valued function

$$m(u,v) := \frac{r}{2}\Big(1 - g(\nabla r, \nabla r)\Big) = \frac{r}{2}\Big(1 + \frac{4\partial_u r \partial_v r}{\Omega^2}\Big), \qquad (11)$$

which coincides with the parameter M in Schwarzschild.

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which coincides with the parameter M in Schwarzschild. Remarkably, the derivatives

$$\partial_u m = 8\pi r^2 \Big(T_{uv}\frac{\partial_u r}{\Omega^2} - T_{uu}\frac{\partial_v r}{\Omega^2}\Big),$$
$$\partial_v m = 8\pi r^2 \Big(T_{uv}\frac{\partial_v r}{\Omega^2} - T_{vv}\frac{\partial_u r}{\Omega^2}\Big),$$

*mmtut
*dcy

are directly controlled in terms of the energy momentum tensor.

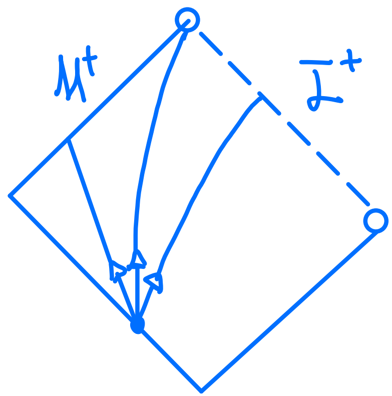
The geodesic flow in Schwarzschild I

The geodesic equations for the null momentum coordinates are given by

$$\begin{cases} \frac{dp^u}{ds} &= \frac{2M}{r^2}(p^u)^2 - \frac{l^2}{2r^3}, \\ \frac{dp^v}{ds} &= -\frac{2M}{r^2}(p^v)^2 + \frac{l^2}{2r^3}, \\ \frac{dl}{ds} &= 0, \end{cases} \tag{12}$$

where $l^2 := r^4\gamma_{AB}p^Ap^B$ ksph is a conserved quantity along the flow, the so-called *angular momentum*. We obtain another conserved quantity along the flow given by the *energy* $E := (1 - \frac{2M}{r})(p^u + p^v)$ *T since Schwarzschild is stationary.

$$\rho = \int_{P_x} f \, d\mathrm{vol}_{P_x}$$



$$\frac{l^2}{E^2} - 27M^2$$

The geodesic flow in Schwarzschild II

The geodesic equation for the radial coordinate is given by

$$\begin{cases} \dot{r} &= p^r, \\ \dot{p}^r &= \dfrac{l^2}{r^4}(r - 3M), \end{cases} \tag{13}$$

*aut, dple

which admits a fixed point corresponding to the unique sphere where null geodesics can orbit, the so-called photon sphere. *unst tropp

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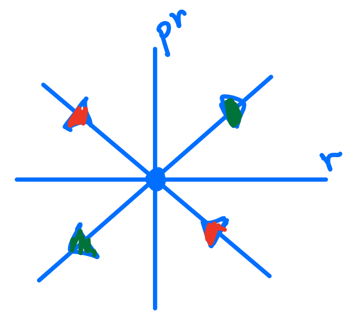
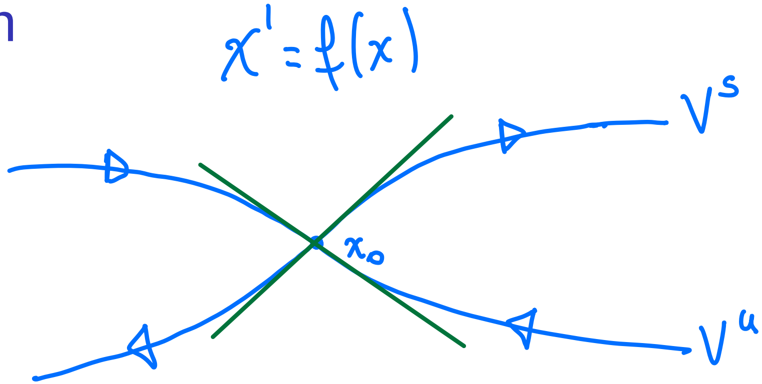
which admits a fixed point corresponding to the unique sphere where null geodesics can orbit, the so-called photon sphere. Linearizing around the fixed point, we obtain the system

$$\begin{cases} \dot{r} &= p^r, \\ \dot{p}^r &= \dfrac{l^2}{81M^4}(r-3M), \end{cases} \tag{14}$$



which admits an hyperbolic fixed point.

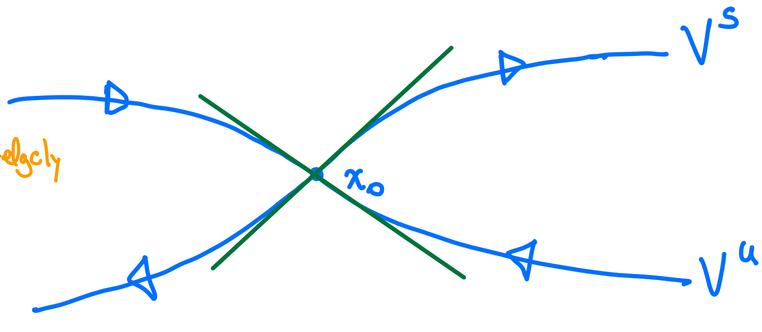
The stable manifold theorem



The stable manifold theorem

$V^s = \{x_0 \in B_1 : x(s) \in B_1 \ \forall s \geq 0\}$ *telgcly

$V^u = \{x_0 \in B_1 : x(s) \in B_1 \ \forall s \leq 0\}$



Theorem (Hadamard-Perron)

Let $f : D \to \mathbb{R}^n$ be a function of class C^k. Let $x_0 \in \mathbb{R}^n$ be an hyperbolic fixed point for the equation

$$x' = f(x)_{\text{*autn}} \tag{15}$$

Then, there exists a neighbourhood B of x_0 such that the sets $V^s \cap B$ and $V^u \cap B$ are <u>*manifolds*</u> *of class C^k containing x_0 and satisfying*

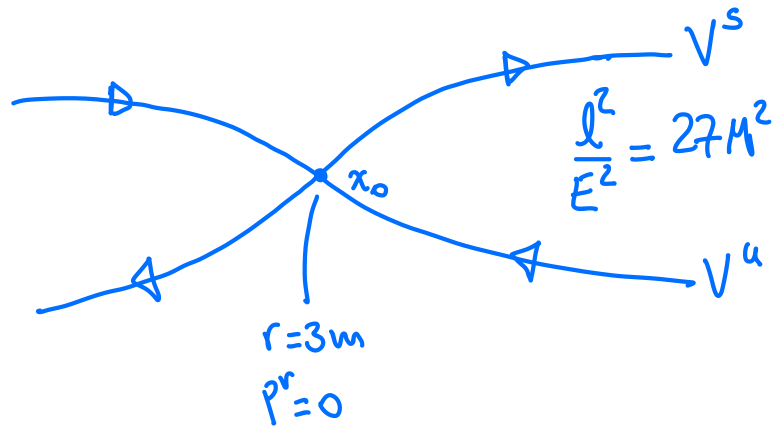
$$T_{x_0}(V^s \cap B) = E^s \quad \text{and} \quad T_{x_0}(V^u \cap B) = E^u. \text{ *Pf} \tag{16}$$

The stable manifold theorem

$$\begin{pmatrix}\dot r\\ \dot p^r\end{pmatrix} = f\begin{pmatrix} r\\ p^r\end{pmatrix}$$

i) codim(B)

ii) Quant. good. control.



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$$T_{x_0}(V^s \cap B) = E^s \quad \textit{and} \quad T_{x_0}(V^u \cap B) = E^u. \tag{16}$$

Decay of the energy momentum tensor

Let us consider a fixed component of the stress energy momentum tensor of matter given by

$$T_{uv}(u,v) = \frac{\pi}{2r^2} \int_{\mathbb{R}+} \int_{\mathbb{R}+} (\Omega^2 p^u)(\Omega^2 p^v) f \frac{dp^v}{p^v} l dl. \quad \times \text{vol} \tag{17}$$

The decay estimates for T_{uv} come from several features of the geodesic flow in Schwarzschild:

1. The red-shift —— $\frac{dp^v}{ds} + \frac{2M}{r^2}(p^v)^2 = \frac{l^2}{2r^3}$
2. Future trapped geodesics —— p^r
3. Decay towards null infinity —— $4\left(1 - \frac{2M}{r}\right)p^u p^v = \frac{l^2}{r^2}$

Derivatives of the energy momentum tensor I

To estimate radial derivatives of the energy momentum tensor like

$$\partial_r T_{uv} = \frac{\pi}{2r^2} \int_{\mathbb{R}+} \int_{\mathbb{R}+} (\Omega^2 p^u)(\Omega^2 p^v) \partial_r f \frac{dp^v}{p^v} l dl + Err,$$

we require bounds for $\partial_r f$. For this purpose we estimate Jacobi fields on the mass-shell.

$$J(t) = \frac{\partial \gamma_\tau}{\partial \tau}(t)\Big|_{\tau=0}, \qquad \nabla_{\dot\gamma} \nabla_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma$$

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Let $V \in T\mathcal{P}$ be an arbitrary vector field on the mass-shell. By the Vlasov equation, we have

$$f(x_0, p_0) = f(x_s, p_s) =: f(\phi_s(x_0, p_0)) \tag{18}$$

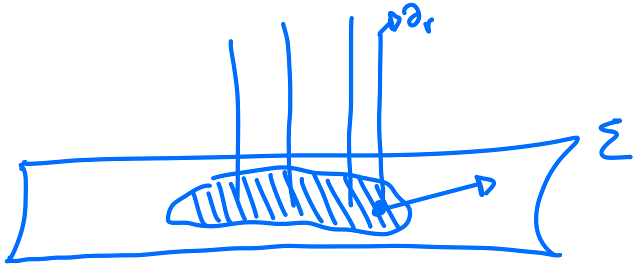
for every point (x_0, p_0) contained on the initial data.

Derivatives of the energy momentum tensor II

As a result, we have

$$V(f)(x_s, p_s) = J(f)(x_0, p_0), \quad \text{J grow} \Rightarrow \partial f \text{ decay} \tag{19}$$

where $J := d\phi_{-s}|_{(x_s,p_s)}(V)$ is a *Jacobi field on the mass shell* along a fixed geodesic γ.



Example: Minkowski

Derivatives of the energy momentum tensor II

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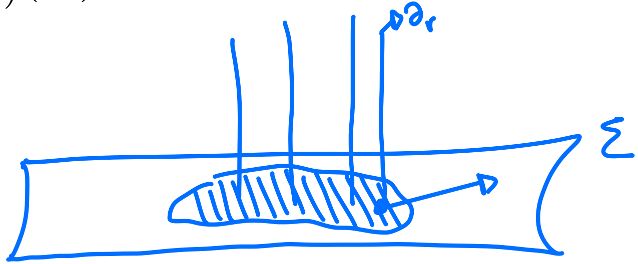

A *Jacobi field on the mass shell* is a vector field along a geodesic which satisfies the so-called *Jacobi equation* given by

$$\widehat{\nabla}_{X_*}\widehat{\nabla}_X J = \widehat{R}(X, J)X, \tag{20}$$

where $\widehat{\nabla}_X$ is the connection over the mass-shell. *geo, Jcb.

Jacobi fields along the photon sphere

Let γ be a null geodesic contained in the equatorial plane of the photon sphere. Then,

$$\nabla_{\dot{\gamma}}\partial_r = \frac{1}{3M}\dot{\gamma}. \tag{21}$$

We are interested in Jacobi fields transversal *flow? to the flow so we work in the quotient of the mass-shell $\mathcal{P}$ by $\mathrm{span}\{\dot{\gamma}\}$.

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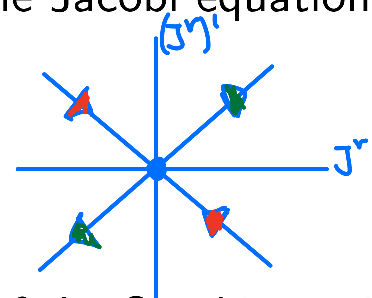
$$\frac{d^2 J^r}{ds^2} = \frac{l^2}{81M^4}{}^{*} J^r(s).$$



A similar computation on the mass-shell in terms of the Sasaki metric shows the same equation for the components J^H and J^V of a radial Jacobi field $J := J^H \mathrm{Hor}(\partial_r) + J^V \mathrm{Ver}(\partial_r)$.

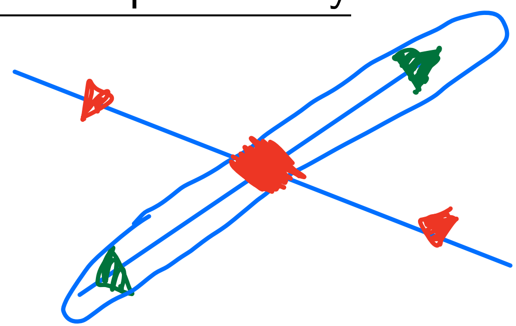
* $\partial_r f$ can grow!

Derivatives of the energy momentum tensor II

Let us investigate the value along the photon sphere of the term

$$\frac{\pi}{2r^2}\int_{\mathbb{R}+}\int_{\mathbb{R}+}(\Omega^2 p^u)(\Omega^2 p^v)Vf\frac{dp^v}{p^v}ldl\bigg|_{r=3m} \tag{22}$$

contained in the derivative $\partial_r T_{uv}$ of the energy momentum tensor. By the computation of Jacobi fields along the photon sphere, we know that Jacobi fields <u>grow or shrink exponentially</u> fast at $\{r = 3m\}$.



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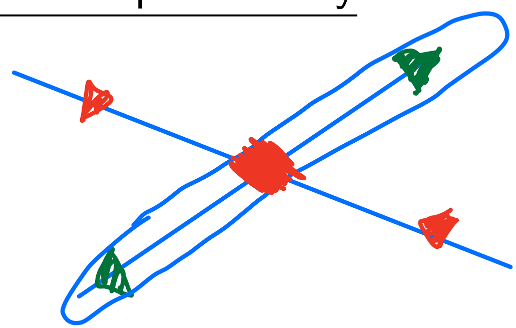
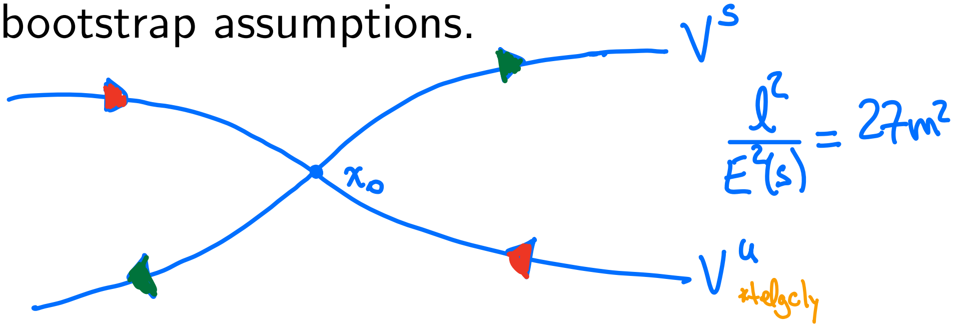


The set of Jacobi fields growing exponentially are <u>concentrated in a small region</u> of $\mathcal{P}_x$! *outcome: ∂T

The nonlinear difficulties I

The result follows via a boostrap argument considering exponential decay of T and ∂T in the bootstrap assumptions.



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Let us focus in the geodesic flow around $\{r = 3m\}$. The geodesic equation for the area radius is given by

$$\begin{cases} \dot{r} &= p^r, \\ \dot{p^r} &= \dfrac{l^2}{r^4}(r - 3m) - 4\pi r\Big(T_{uu}(p^u)^2 - 2T_{uv}p^u p^v + T_{vv}(p^v)^2\Big)_{*}, \end{cases} \tag{23}$$

where $l^2 := r^4 \gamma_{AB} p^A p^B_{\#sph}$ is the angular momentum of a geodesic and $m(u, v)$ is the Hawking mass.

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where $l^2 := r^4\gamma_{AB}p^A p^B$ is the angular momentum of a geodesic and $m(u,v)$ is the Hawking mass. Although, T is not Killing anymore, we can still work with the *energy of a geodesic* γ

$$E(s) := -g(T,\dot{\gamma}) = -\partial_u r p^u(s) + \partial_v r p^v(s). \tag{24}$$

The nonlinear difficulties II

Remarkably, the derivative of the energy satisfies

$$\frac{dE}{ds} = 4\pi r\Big((p^u)^2 T_{uu} - (p^v)^2 T_{vv}\Big). \tag{25}$$

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However, we still need estimates for the Jacobi fields around $\{r = 3m\}$.

Mimicking the previous computation in Schwarzschild, we have

$$\begin{aligned}\nabla_{\dot\gamma}\partial_r = {} & \frac{\Omega^2 m}{2r^2}\Big[\frac{p^v}{(\partial_u r)^2}\partial_u + \frac{p^u}{(\partial_v r)^2}\partial_v\Big] + \frac{2}{r}(p^\phi\partial_\phi + p^\theta\partial_\theta) \\ & + 4\pi r\Big[\frac{p^u T_{uu} - p^v T_{uv}}{(\partial_u r)^2}\partial_u + \frac{p^v T_{vv} - p^u T_{uv}}{(\partial_v r)^2}\partial_v\Big],\end{aligned}$$

which has many error terms. Several more error terms come out when studying Jacobi fields on the mass shell.

The nonlinear difficulties III

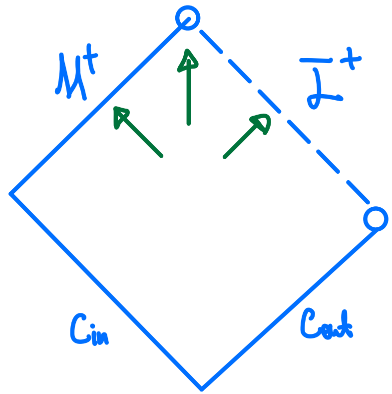
Furthermore, for every future trapped geodesic there are Jacobi fields for which

$$\frac{d^2 J^r}{ds^2} = \frac{l^2}{81m^4} J^r(s) + Err$$

around $r = 3m$. Similarly, for Jacobi fields on the mass shell.

We do not go into further details on the errors contained in the Jacobi equation, however, we find several terms involving T and ∂T where the bootstrap assumptions come into place.

Asymptotic stability of Schwarzschild



Theorem (V.)

The exterior of the Schwarzschild family is asymptotically stable as a solution of the spherically symmetric Einstein–massless Vlasov system. More precisely, for every initial data sufficiently close to Schwarzschild, the resulting solution asymptotes exponentially to another member of the Schwarzschild family.

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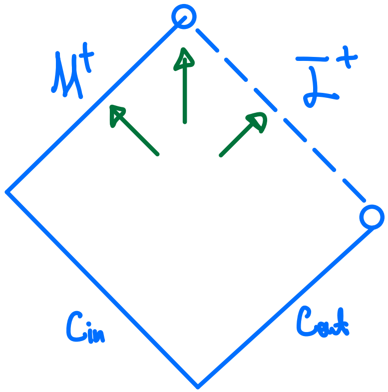

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Thank you for your attention!